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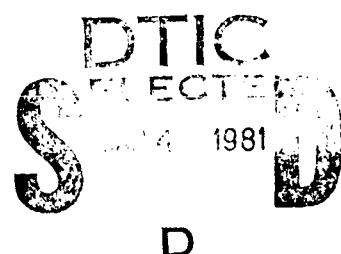ON THE E-OPTIMALITY
OF CERTAIN PBIB DESIGNS

by

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It is shown that several families of PBIB designs are E-optimal over the collection of all block designs. Among these, the partial geometries with two associate classes; PBIB designs with $\lambda_1 = 1, \lambda_2 = 0$ and fewer blocks than varities; PBIB designs with triangular schemes of size n, $\lambda_1 = 0$ $\lambda_2 = 1$ and block size $k \geq n/2$ (or $\lambda_1 - 1, \lambda_2 = 0$ and $k \geq n-1$); PBIB designs with L_1 schemes based on v varieties with $\lambda_1 = 0$ $\lambda_2 = 1$, $k \geq \sqrt{v}$ (or $\lambda_1 - 1, \lambda_2 = 0$ and either $i - 1 \leq \sqrt{v} \leq k$ or $k \leq \sqrt{v} \leq i - 1$). The duals of these designs are also E-optimal. When uniform on rows, these designs remain E-optimal in the additive setting of two way

elimentation of heterogeneity.

ON THE E-OPTIMALITY
OF CERTAIN PBIB DESIGNS

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ABSTRACT

It is shown that several families of PBIB designs are E-optimal over the collection of all block designs. Among these, the partial geometries with two associate classes are shown, PBIB designs with $\lambda_1 = 1$, $\lambda_2 = 0$ and fewer blocks than varieties; PBIB designs with triangular schemes of size n, $\lambda_1 = 0$, $\lambda_2 = 1$ and block size $k \geq \frac{n}{2}$ (or $\lambda_1 = 1$, $\lambda_2 = 0$ and $k \geq n - 1$); PBIB designs with L_i schemes based on v varieties with $\lambda_1 = 0$, $\lambda_2 = 1$, $k \geq \sqrt{v}$ (or $\lambda_1 = 1$, $\lambda_2 = 0$ and either $i - 1 \leq \sqrt{v} \leq k$ or $k \leq \sqrt{v} \leq i - 1$). The duals of these designs are also E-optimal. When uniform on rows, these designs remain E-optimal in the additive setting of two way elimination of heterogeneity.

1. Introduction and notation

The object of this work is to investigate the E-optimality of discrete statistical experiments in the additive setting of one and two way elimination of heterogeneity. For v , b , k positive integers, we denote by $\Omega_{v,b,k}$ the collection of all $k \times b$ arrays with varieties $1, 2, \ldots, v$ as entries ($2 \leq k < v$). Any such array $d \in \Omega_{v,b,k}$ is called a design. The columns of d are called blocks. A design is said to be binary if each block of d consists of distinct varieties; d is called equireplicated if each variety occurs the same numbers of times throughout the whole array d . If each variety appears the same number of times in each row of d , d is called uniform.

Let α_i be the unknown effect of variety i and β_j be the (unknown) effect of the jth block. In the additive model of elimination of heterogeneity in one direction, we assume that the expectation of an observation on variety i in the jth block of d is $\alpha_i + \beta_j$.

If, in addition, γ_m denotes the (unknown) effect of the mth row of d , and there is evidence of row effects in the observations, we shall assume that the expected value of an observation on variety i in block j and row m of d is $\alpha_i + \beta_j + \gamma_m$.

Under both models we assume the kb observations uncorrelated, with common (unknown) variance σ^2 . The main interest is in comparing the variety effects α_1 , $\alpha_2, \ldots, \alpha_v$.

The information matrices of variety effects under the two models are, respectively

$$kC_d = k \, \text{diag}(r_{d1}, \ldots, r_{dv}) - N_d N_d'$$

and

$$k\tilde{C}_d = k \, \text{diag}(r_{d1}, \ldots, r_{dv}) - N_d N_d' - b^{-1} M_d (kI-J) M_d'$$

where $N_d = (n_{dij})$, $M_d = (m_{dij})$, with n_{dij} (resp. m_{dij}) indicating the number of times variety i appears in the jth block (resp. row) of d; r_{di} is the replication number of variety i in d. J denotes the matrix with all its entries 1 and I is the identity matrix. By λ_{dij} we denote the $(i,j)^{th}$ entry of $N_d N_d'$. It is known that for any d both C_d and $\tilde{C}_d$ are nonnegative definite with row sums zero. Let further $0 = \mu_{d0} \leq \mu_{d1} \leq \ldots \leq \mu_{d,v-1}$ and $0 = \tilde{\mu}_{d0} \leq \tilde{\mu}_{d1} \leq \ldots \leq \tilde{\mu}_{d,v-1}$ be the eigenvalues of C_d and $\tilde{C}_d$, respectively.

A design d^* is called E-optimal over $\Omega_{v,b,k}$ (under a given model) if the maximal variance of normalized best linear unbiased estimators of estimable functions is minimal under d^*. It is well-known (Ehrenfeld (1955); see also Kiefer (1959) and the beautiful expository article of Kiefer (1978)) that d^* is E-optimal over $\Omega_{v,b,k}$ under the one way elimination (resp. two way elimination) if and only if $\mu_{d^*1} \geq \mu_{d1}$ (resp. $\tilde{\mu}_{d^*1} \geq \tilde{\mu}_{d1}$) for all $d \in \Omega_{v,b,k}$. In short, E-optimality deals with the association

$$d \to C_d \to \mu_{d1}$$

and the objective of finding the design d with maximal μ_{d1}.

The experimental setting which involves relatively few blocks (say $0 < b < 2v$) is of notable practical importance. Just to focus attention, suppose we have $v = 15$ varieties,

$b = 15$ blocks, $k = 3$ varieties per block and that there is evidence of an additive variety and block response. The experimenter would like to use a design $d°$ which (if not optimal) is efficient in the E-sense, i.e., $\mu_{d°1} \geq \mu_{d1}$ for most d. You would then say, well, maybe this $d°$ will do:

$$
d°: \quad
\begin{matrix}
1 & 1 & 8 & 8 & 12 & 14 & 1 & 13 & 13 & 9 & 9 & 11 & 2 & 13 & 15 \\
14 & 10 & 10 & 4 & 10 & 4 & 6 & 2 & 7 & 7 & 3 & 7 & 3 & 5 & 12 \\
8 & 4 & 6 & 12 & 14 & 6 & 15 & 9 & 3 & 5 & 11 & 2 & 5 & 15 & 11
\end{matrix}
\quad (1.1)
$$

It's connected, it's binary, it's equireplicated, and a pair of distinct varieties appears in at most one block. Looks good, as it is $M - S$ optimal, (see Eccleston and Hedayat (1974) or Shah (1960)) and surely does not favor any variety (in the senses mentioned above). Only trouble is, $d°$ is in fact a very inefficient design in the E-sense.

With a bit of luck you would have probably recommended:

$$
d^*: \quad
\begin{matrix}
1 & 1 & 1 & 2 & 2 & 2 & 5 & 5 & 5 & 8 & 8 & 9 & 9 & 7 & 6 \\
4 & 10 & 3 & 4 & 3 & 10 & 4 & 3 & 10 & 7 & 12 & 6 & 11 & 11 & 14 \\
7 & 13 & 6 & 9 & 8 & 14 & 12 & 11 & 15 & 15 & 13 & 15 & 13 & 14 & 12
\end{matrix}
\quad (1.2)
$$

which has features very similar to $d°$ in the $M - S$ sense (but luckily not in the E-sense).

However, one should hope that you haven't sent him back with such a design as

$$
d: \quad
\begin{matrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 10 & 11 & 13 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 10 & 11 & 13 & 15 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 4 & 12 & 14 & 8
\end{matrix}
\quad (1.3)
$$

The surprizing fact is that d is E-better than d° and the discrepancy in the E-performance between d^* and d° is by a factor of more than 8. The exact figures are $3\mu_{d^\circ 1} = .571$, $3\mu_{d1} = .625$ and $3\mu_{d^*1} = 5$.

We would like to stress two things. That this is not an isolated example (not when the number of blocks is small); and that $M - S$ optimal designs with large number of blocks do not differ in the E-performance in such a surprizing way. In the former case a way to avoid bad $M - S$ optimal designs is suggested in Constantine (1980).

In the next section we show that a design $d^* \in \Omega_{v,b,k}$ with $\mu_{d^*1} \geq \frac{v}{v-k}(r-1)(k-1)$ and $r = bkv^{-1}$ integral, is E-optimal over all block designs. This result has a number of consequences in settings with b small. Various families of PBIB designs (mentioned below) with λ_1 and λ_2 zero or one are proved E-optimal over all block designs. The partial geometries with two associate classes are such instances. Among the triangular PBIB designs with schemes of size n: those with $\lambda_1 = 0$, $\lambda_2 = 1$, $k \geq \frac{n}{2}$ and those with $\lambda_1 = 1$, $\lambda_2 = 0$ and $k \geq n - 1$. Among the ones with L_i schemes: those with $\lambda_1 = 0$, $\lambda_2 = 1$, $k \geq \sqrt{v}$ and those with $\lambda_1 = 1$, $\lambda_2 = 0$ and either $i - 1 \leq \sqrt{v} \leq k$ or $k \leq \sqrt{v} \leq i - 1$. The duals of these designs are E-optimal as well. These E-optimality results extend easily to the setting of two way elimination of heterogeneity when the above designs are also uniform.

2. Results

We denote by $\underset{\sim}{1}$ the column vector with all its entries 1

and by J the (not necessarily square) matrix with all its entries 1. The following two lemmas provide upper bounds for μ_{d1}. Various bounds can also be found in Chakrabarti (1963), Jacroux (1980), Cheng (1980) or Constantine (1979b).

Lemma 2.1: Let C be a $v \times v$ nonnegative definite matrix with zero row and column sums. Denote the eigenvalues of C by $0 = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_{v-1}$. Then the sum of entries in any $m \times m$ principal minor of C is at least $\frac{m(v-m)}{v} \mu_1$; $(1 \leq m \leq v - 1)$.

Proof: Observe that a matrix obtained form C by row and (same) column permutations has the same eigenvalues as C. It will therefore be enough to prove the lemma for the $m \times m$ leading principal minor of C. Call this leading principal minor M. Then

$$\underset{\sim}{1}' M \underset{\sim}{1} = \left(\begin{pmatrix} \frac{1}{0} \end{pmatrix} - \frac{m}{v} \underset{\sim}{1} \right)' C \left(\begin{pmatrix} \frac{1}{0} \end{pmatrix} - \frac{m}{v} \underset{\sim}{1} \right) \geq$$

$$\left(\begin{pmatrix} \frac{1}{0} \end{pmatrix} - \frac{m}{v} \underset{\sim}{1} \right)' \left(\begin{pmatrix} \frac{1}{0} \end{pmatrix} - \frac{m}{v} \underset{\sim}{1} \right) \mu_1 = \frac{m(v-m)}{v} \mu_1$$

as stated. The inequality relies on the known fact that

$$\mu_1 = \min_{x' \underset{\sim}{1} = 0} \frac{x' C x}{x' x}$$

and on observing that $\left(\begin{pmatrix} \frac{1}{0} \end{pmatrix} - \frac{m}{v} \underset{\sim}{1} \right)' \underset{\sim}{1} = 0$ (since the $\underset{\sim}{1}$ in $\begin{pmatrix} \frac{1}{0} \end{pmatrix}$ is $m \times 1$). This ends the proof.

Our next lemma gives an upper bound for μ_{d1} when $d \in \Omega_{v,b,k}$ is equireplicated.

Lemma 2.2: If an equireplicated design $d \in \Omega_{v,b,k}$ contains a block which consists of m distinct varieties $(2 \leq m \leq k)$,

then

$$k\mu_{d1} \le \frac{v}{m(v-m)} (k-1)(mr-k) .$$

Proof: By eventually relabeling the varieties and reshuffling the blocks, we can assume that the first block in d consists of n_{d11} 1's , n_{d21} 2's ... and n_{dm1} m's . Index the rows and columns of C_d by the varieties $1, 2,...,v$ (in this order), and let M_d be the $m \times m$ leading principal minor of C_d . Observe, firstly, that $\sum_{j=1}^{b} n_{dij} = r$ and that $\sum_{j=1}^{b} n_{dij}^2 \ge n_{di1} + \sum_{j=2}^{b} n_{dij}^2 \ge \sum_{j=1}^{b} n_{dij} = r$. Hence $\sum_{j=2}^{b} n_{dij}^2 \ge r - n_{di1}$ and therefore $\sum_{j=1}^{b} n_{dij}^2 \ge n_{di1}^2 + r - n_{di1}$. Secondly, note that $\sum_{i \ne j}^{m} \lambda_{dij}$ (a sum of $m(m-1)$ nonnegative terms) satisfies $\sum_{i \ne j}^{m} \lambda_{dij} = \sum_{i \ne j}^{m} \sum_{u=1}^{b} n_{diu} n_{dju} \ge \sum_{i \ne j}^{m} n_{di1} n_{dj1}$. Using these two inequalities and the fact that $\sum_{i=1}^{m} n_{di1} = k$, we obtain:

$$\underset{\sim}{1}' Md \underset{\sim}{1} = mrk - \sum_{i=1}^{m} \sum_{j=1}^{b} n_{dij}^2 - \sum_{i \ne j} \lambda_{dij}$$

$$\le mrk - \sum_{i=1}^{m} (n_{di1}^2 + r - n_{di1}) - \sum_{i \ne j}^{m} n_{di1} n_{dj1}$$

$$= mrk - (\sum_{i=1}^{m} n_{di1})^2 - mr + \sum_{i=1}^{m} n_{di1}$$

$$= (k-1)(mr - k)$$

That $k\mu_{d1} \le \frac{v}{m(v-m)} (k-1)(mr - k)$ follows now from Lemma 2.1. This ends the proof.

Through the remainder of the paper, let the varieties in a design $d \in \Omega_{v,b,k}$ be always labeled so that the replication

numbers r_{di} satisfy $r_{d1} \leq r_{d2} \leq \ldots \leq r_{dv}$. We are now ready to prove our first result.

<u>Theorem 2.1</u>: Let $r = bkv^{-1}$ be an integer. A design $d^* \in \Omega_{v,b,k}$ which satisfies $k\mu_{d*1} \geq \frac{v}{v-k} (r-1)(k-1)$ is E-optimal over all block designs.

<u>Proof</u>: Let d be any design in $\Omega_{v,b,k}$. Then d is either equireplicated or it is not. Suppose it is not. Then $r_{d1} \leq r - 1$ and by Lemma 2.1 with $m = 1$ we have:

$$k\mu_{d1} \leq \frac{v}{v-1} r_{d1}(k-1) \leq \frac{v}{v-1} (r-1)(k-1) < \frac{v}{v-k}(r-1)(k-1) \leq k\mu_{d*1}$$

which show that such a design is strictly E-worse than d^* .

Assume now that d is equireplicated. We may also assume that d has a block which consists of m distinct varieties $(2 \leq m \leq k)$. (Observe that if d has no such block, the information matrix C_d is then the zero matrix, and hence for such d we have $\mu_{d1} = 0 < \mu_{d*1}$.) By Lemma 2.2 we can write

$$k\mu_{d1} \leq \frac{v}{m(v-m)} (k-1)(mr - k) .$$

Let $Q(m) = -k\mu_{d*1}m^2 + (vk\mu_{d*1} - v(k-1)r)m + vk(k-1)$. Note that

$$\frac{v}{m(v-m)}(k-1)(mr - k) \leq k\mu_{d*1} , \text{ for all } 2 \leq m \leq k$$

if and only if $Q(m) \geq 0$, for all $2 \leq m \leq k$. Since $Q(m)$ is a quadratic in m with negative leading coefficient and $Q(0) = vk(k-1) > 0$ checking that $Q(k) \geq 0$ would insure that $Q(m) \geq 0$ for all $2 \leq m \leq k$. By assumption

$k\mu_{d*1} \geq \frac{v}{v-k}(r-1)(k-1)$ which implies $-k^2\mu_{d*1} + vk\mu_{d*1} - v(k-1)r + v(k-1) \geq 0$. In terms of Q this last inequality is simply $k^{-1}Q(k) \geq 0$. Since k is positive it follows that $Q(k) \geq 0$, as desired. We have therefore shown

$$k\mu_{d1} \leq \frac{v}{m(v-m)}(k-1)(mr-k) \leq k\mu_{d*1} , \text{ for all } 2 \leq m \leq k .$$

This concludes the proof.

The following aspect of Theorem 2.1 should perhaps be stressed. It is easy to show that for two (real) matrices A and B , the products AB and BA have the same set of eigenvalues. Given an equireplicated design $d \in \Omega_{v,b,k}$ (with $r = bkv^{-1}$ integral), we call $\delta \in \Omega_{b,v,r}$ the design dual to d , if $N_\delta = N_d'$. Hence $rC_\delta = rkI - N_d'N_d$. By the above remark it directly follows that kC_d and rC_δ have the same set of eigenvalues. Thus, if $d* \in \Omega_{v,b,k}$ satisfies the assumption of Theorem 2.1, then

$$r\mu_{\delta*1} = k\mu_{d*1} \geq \frac{v}{v-k}(r-1)(k-1) = \frac{b}{b-r}(k-1)(r-1) ,$$

and hence $\delta* \in \Omega_{b,v,r}$ satisfies also the assumption of the theorem. This gives

<u>Corollary 2.1</u>   Let $r = bkv^{-1}$ be an integer. The dual of a design $d* \in \Omega_{v,b,k}$ with $\mu_{d*1} \geq \frac{v}{v-k}(r-1)(k-1)$ is E-optimal over $\Omega_{b,v,r}$.

All the designs which we shall prove E-optimal next, have E-optimal duals, in view of this corollary.

Although not apparent from the statement of Theorem 2.1, the lower bound $\frac{v}{v-k}(r-1)(k-1)$ is useful for designs with relatively small number of blocks $(0 < b < 2v$, say), and

hardly ever otherwise. Let us examine a few consequences
of Theorem 2.1.

Theorem 2.2   A Partially Balanced Incomplete Block design
based on v varieties and b blocks of size k , with in-
tegral parameters as below, is E-optimal over all block de-
signs: $r = bkv^{-1}$, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = bk(k-1)v^{-1}$,
$n_2 = v - 1 - bk(k-1)v^{-1}$, $t = k(k-1)(bk-v)v^{-1}(v-k)^{-1}$,
$p_{11}^1 = k - 2 + [k(k-1)(bk-v) - v(v-k)](bk-v)v^{-2}(v-k)^{-1}$,
$p_{11}^2 = bk^2(k-1)(bk-v)v^{-2}(v-k)^{-1}$.

Proof: Let a PBIB design with parameters as above be denoted
by d* . Then C_{d*} is known to have two distinct nonzero
eigenvalues $0 < \mu_{d*1} < \mu_{d*2}$, where $k\mu_{d*1} = r(k-1) - k + t + 1$.
For a proof see Connor and Clatworthy (1954), Bose and Mesner
(1959) or Raghavarao (1971, p.195). It is straightforward to
check that

$$r(k-1) - k + t + 1 = k\mu_{d*1} = \frac{v}{v-k}(r-1)(k-1) .$$

We are then done by Theorem 2.1.

The partial geometries (r,k,t) with two associate
classes, defined by Bose (1963), satisfy the conditions of
Theorem 2.2. Whence,

Corollary 2.2. Partial geometries with two associate classes
are E-optimal over all block designs.

For a thorough reading on classical finite geometries we
refer the reader to the book of Dembowski (1968). Many PBIB
designs can be constructed from partial geometries. The

varieties would be the points and the blocks would be the lines of the geometry. We mention the work of Seiden (1961) and Ray-Chaudhuri (1962) in connection with this. The design (1.2) is an example of a partial geometry of the simplectic type.

Bose and Clatworthy (1955) showed that PBIB designs with $b < v$, $\lambda_1 = 1$ and $\lambda_2 = 0$ necessarily have parameters as those listed in Theorem 2.2. We therefore have:

<u>Corollary 2.3</u>. A Partially Balanced Incomplete Block design with $b < v$, $\lambda_1 = 1$ and $\lambda_2 = 0$ is E-optimal over all block designs.

Connor and Clatworthy (1954) found the nonzero eigenvalues of the information matrix of a PBIB design with two associate classes to be

$$k\mu_1 = r(k-1) + \tfrac{1}{2}[(\lambda_1 - \lambda_2)(-\gamma + \sqrt{\Delta}) + \lambda_1 + \lambda_2]$$

and

$$k\mu_2 = r(k-1) + \tfrac{1}{2}[(\lambda_1 - \lambda_2)(-\gamma - \sqrt{\Delta}) + \lambda_1 + \lambda_2)] \ .$$

It is easy to see that $\mu_1 < \mu_2$ if and only if $\lambda_1 < \lambda_2$. γ and Δ are expressed in terms of the parameters of the association scheme as $\gamma = p_{12}^2 - p_{12}^1$ and $\Delta = (p_{12}^2 - p_{12}^1)^2 + 2(p_{12}^1 + p_{12}^2) + 1$ (See Raghavarao (1971, p. 126)).

We now prove the following:

<u>Theorem 2.3</u> (a) A Partially Balanced Incomplete Block design with $\lambda_1 = 0$, $\lambda_2 = 1$ and $\gamma - \sqrt{\Delta} + 1 \geq \dfrac{2(k-1)(rk-v)}{v-k}$ is E-optimal over all block designs; (b) A Partially Balanced Incomplete Block Design with $\lambda_1 = 1$, $\lambda_2 = 0$ and $1 - \gamma - \sqrt{\Delta} \geq \dfrac{2(k-1)(rk-v)}{v-k}$

is E-optimal over all block designs.

Proof: Let d^* be a PBIB design as in (a). Then
$\gamma - \sqrt{\Delta} + 1 \geq \dfrac{2(k-1)(rk-v)}{v-k}$ implies $(v-k)(\gamma - \sqrt{\Delta} + 1)$
$\geq 2rk^2 - 2vk - 2rk + 2v$ which, in turn, implies

$$r(k-1)(v-k) + \frac{1}{2}(v-k)(\gamma - \sqrt{\Delta} + 1) \geq v(r-1)(k-1)$$

and which can be rewritten as

$$r(k-1) + \frac{1}{2}(\gamma - \sqrt{\Delta} + 1) \geq \frac{v}{v-k}(r-1)(k-1) \ .$$

We are now done by Theorem 2.1, since it follows from the
paragraph preceding the statement of Theorem 2.3 that
$k\mu_{d^*1} = r(k-1) + \frac{1}{2}(\gamma - \sqrt{\Delta} + 1)$. As for part (b), it can be
similarly proved.

When specialized to various known association schemes
Theorem 2.3 yields several corollaries:

Corollary 2.4: A Partially Balanced Incomplete Block design
with a triangular association scheme of size n , $\lambda_1 = 0$,
$\lambda_2 = 1$ and block size $k \geq \frac{n}{2} \geq 3$ is E-optimal over all
block designs.

Proof: In this case $\gamma = n - 5$ and $\Delta = (n-2)^2$, $v = \dfrac{n(n-1)}{2}$
and $r(k-1) = \dfrac{(n-2)(n-3)}{2}$. Now, for $n \geq 6$ we have the fol-
lowing chain of implications: $k \geq \frac{n}{2} \Rightarrow k \geq \dfrac{2v}{v-r(k-1) + 1} \Rightarrow$
$2v \leq (v-r(k-1) + 1)k \Rightarrow k - v \geq r(k-1)k - v(k-1) \Rightarrow$
$\gamma - \sqrt{\Delta} + 1 = -2 \geq \dfrac{2(k-1)(rk-v)}{v-k}$, as desired. The proof can
now be concluded by part (a) of Theorem 2.3.

The family of PBIB designs mentioned by Shrikhande (1965)
satisfies the assumptions of Corollary 2.4. These PBIB designs

are therefore E-optimal over all block designs (with the same v, b and k).

In a similar way, and with the help of Theorem 2.3(b), we can prove

Corollary 2.5. A Partially Balanced Incomplete Block design with a triangular association scheme of size n , $\lambda_1 = 1$, $\lambda_2 = 0$ and block size $k \geq n - 1$ is E-optimal over all block designs.

Examples of such families can be found in Masuyama (1965). Some can be obtained, for example, by just writing the rows of the triangular association scheme as the blocks of the design.

We now turn to PBIB designs with L_i schemes. Here $v = s^2$ (for some integer $s \geq 2$) , $\gamma = s - 2i + 1$ and $\Delta = s^2$. In much the same way as Corollary 2.4 was obtained, we are led to

Corollary 2.6. A Partially Balanced Incomplete Block design with an L_i association scheme, $\lambda_1 = 0$, $\lambda_2 = 1$ and block size $k \geq \sqrt{v}$ is E-optimal over all block designs.

When $\sqrt{v}$ is a prime or a prime power, Clatworthy (1967) gives a class of PBIB designs with the following parameters:

$$v = s^2 , b = s(s + 1 - i) , k = s , r = s + 1 - i , \lambda_1 = 0 \text{ and } \lambda_2 = 1 .$$

By our corollary, all these designs are E-optimal.

Let us now derive:

Corollary 2.7. A Partially Balanced Incomplete Block design with an L_i association scheme, $\lambda_1 = 1$, $\lambda_2 = 0$ and the

block size k satisfying either $i - 1 \le \sqrt{v} \le k$ or $k \le \sqrt{v} \le i - 1$ is E-optimal over all block designs.

<u>Proof</u>: With γ and Δ as in the paragraph preceding Corollary 2.6 we have $-\gamma - \sqrt{\Delta} + 1 = 2(i-s)$ and $r(k-1) = is - i$. The condition $i - 1 \le s \le k$ or $k \le s \le i - 1$ can be written as $(s - i + 1)(k - s) \ge 0$. Upon multiplying by s and expending, it leads to $vi - vs + ks + vk - v - kis \ge 0$. Using the fact that $r(k-1) = is - i$ this gives $(v-k)(i-s) \ge (k-1)(rk-v)$ or

$$- \gamma - \sqrt{\Delta} + 1 = 2(i-s) \ge \frac{2(k-1)(rk-v)}{v-k} \quad .$$

With this last condition satisfied, we are done by Theorem 2.3(b).

As an example of PBIB designs whose parameters satisfy the assumptions of Corollary 2.7 we mention the ones with

$$v = s^2, \ b = is, \ k = s, \ r = i, \ \lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 0 ; \ (s \ge 2) \quad .$$

These designs appear in Clatworthy (1967).

Let us now turn our attention to the setting of two way elimination. Note that the matrix $C_d - \tilde{C}_d = k^{-1}b^{-1}M_d(kI - J)M_d'$ is nonnegative definite. Hence (see Bellman (1979)) $\tilde{\mu}_{di} \le \mu_{di}$, $1 \le i \le v - 1$. If $d*$ is a uniform design, then $M_{d*} = bv^{-1}J$ and it is easily seen that $\tilde{C}_{d*} = C_{d*}$. Moreover, if such a uniform design $d*$ is E-optimal in the setting of one way elimination, then for any other design d we have

$$\tilde{\mu}_{d1} \le \mu_{d1} \le \mu_{d*1} = \tilde{\mu}_{d*1}$$

and hence $d*$ is also E-optimal in the setting of two way

elimination. This simple argument extends to $D-$, $A-$, type I criteria (Cheng (1978)) and more generally to Schur-optimality (Constantine (1979a). Our observation can be formulated as

Lemma 2.3: A uniform E-optimal design in the setting of one way elimination is also E-optimal in the setting of two way elimination.

The results obtained so far do then extend in the following way:

Corollary 2.8: If any of the designs in our previous theorems and corollaries are uniform, they are then E-optimal for the settings of both one and two way elimination of heterogeneity.

It is often possible to rearrange the varieties in each individual block and thus achieve uniformity of the design. As an example, the partial geometry (1.2) can be written as

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
d**:	3	8	11	1	15	9	14	12	4	2	13	5	10	6	7	(2.1)
	6	3	5	7	10	15	11	13	2	14	9	4	1	12	8	

Let us remind the reader that if there is no evidence of row effects in the observations, then $d*$ and $d**$ are equally efficient to use. But if row effects are truly present, the E-performances of $d*$ and $d**$ are, respectively, $3\tilde{\mu}_{d*1} = .7226$ and $3\tilde{\mu}_{d**1} = 5$. Trusting that the response is indeed additive, $d**$ is then clearly to be preferred.

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